Thurstonian models as Generalized Linear Models

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Outline

1. General and Generalized linear models
2. Triangle, duo-trio and K-AFC as GLIMs
3. The R language.
4. A-Not A and Same-different methods
5. Replicated difference tests
6. Summary

The (General) Linear Model

$$y = X\beta + \varepsilon$$

or

$$y_i = \mathbf{x}_i\beta + \varepsilon_i$$

- T-tests
- Analysis of variance (ANOVA, fixed effects)
- Simple and Multiple regression analysis (MLR)
- Analysis of covariance (ANCOVA)

The (General) Linear Model: Results

1. The estimated parameter values: $\hat{\beta}$

2. The estimated uncertainty about these: $Cov(\hat{\beta})$

1. Everything else is deduced from this!

2. We have explicit formulas for computation!

Fictitious example

40 male and 40 female consumers rated sweetness intensity (y) of samples with varying stimulus concentrations (x).

A linear model with a gender-dependent intensity-stimulus relation:

$$y_i = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i + \varepsilon_i$$

OR EQUIVALENTLY:

$$E(y_i) = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i$$

Fictitious example:
What if we had 80 triangle test instead?

$$y_i \sim bin(1, p_i), i = 1,\ldots,80.$$

First approach:

$$y_i = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i + \varepsilon_i$$

ANY REAL NUMBER

0 or 1

$$\varepsilon_i \sim N(0, \sigma^2)$$

Not natural!

Fictitious example: What if we had 80 triangle test instead?

$$y_i \sim bin(1, p_i), i = 1,\ldots,80.$$

Second approach:

$$Ey_i = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i$$

Fictitious example:
What if we had 80 triangle test instead?

$$y_i \sim bin(1, p_i), i = 1,\ldots,80.$$

Second approach:

$$p_i = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i$$

ANY REAL NUMBER

Any number between 0 and 1

Better, but not perfect!!

Fictitious example: What if we had 80 triangle tests instead?

$$y_i \sim bin(1, p_i), i = 1,\ldots,80.$$

Final approach:

$$\log(p_i/(1-p_i)) = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i$$

Logistic Regression Model:

- y is binomial *distributed*
- E(y) is *linked* to a linear model by the logit (log-odds) function

The Generalized Linear Model (GLIM) family

1. The *distribution* of y
   (binomial, poisson, gamma, normal etc)
2. The *link* between E(y) and the linear model structure
   (logit, probit, log, inverse etc)

The Logistic Regression Model is a GLIM with

1. binomial *distribution*
2. logit *link*

A theoretical AND computational/practical framework for statistical analysis in many situations!

The Generalized Linear Model: Results

1. The estimated parameter values: $\hat{\beta}$

2. The estimated uncertainty about these: $Cov(\hat{\beta})$

1. Everything else is deduced from this!

2. We have simple iterative methods and available software for computation!

Triangle, duo-trio and K-AFC as GLIMs

Find d-prime from proper psychometric function g:

$$g(d') = \hat{p}$$

Corresponding model:

$$y_i \sim bin(1, p_i) \quad \textit{(Distribution)}$$

$$g^{-1}(p_i) = \delta \quad \textit{(Link)}$$

Triangle, duo-trio and K-AFC as GLIMs

A practical perspective:

1. Find some GLIM-software

2. Define the psychometric link functions in the software (user defined links are usually an option)

3. Do a ”usual linear type” (GLIM) statistical analysis with an inbuilt routine to obtain:

   1. Estimates of d-primes
   2. Variances of d-primes
   3. Comparisons of different d-primes

Software

http://www.r-project.org/

A language and environment for
statistical computing and graphics

Similar to the S-language (Splus)

Free Software

Define the psychometric link functions in the software

- package provided by me along with this paper!

For instance, 3-AFC:

```
threeAFCg<-function(x,d) dnorm(x-d)*pnorm(x)^2
threeAFCgd<-function(x,d) -(x-d)*dnorm(x-d)*pnorm(x)^2

threeAFC<-binomial()
threeAFC$link <- "Link for the 3-AFC test"
threeAFC$linkinv <- function(eta) integrate(threeAFCg,-Inf,Inf,d=eta)$value
threeAFC$mu.eta <- function(eta) integrate(threeAFCgd,-Inf,Inf,d=eta)$value
threeAFCg2<-function(d,p) -p+threeAFC$linkinv(d)
threeAFC$linkfun <- function(mu){
if (mu>1/3) res<- uniroot(threeAFCg2,c(0,10),p=mu)$root
if (mu<=1/3)res <- 0
res
}
```

Do a usual statistical (GLIM) analysis!

Example:
10 out of 15 correct answers in a 3-AFC test:

```
glm(t(c(10,5))~1, family=threeAFC,mustart=10/15)

                Estimate    Std. Error  z value      Pr(>|z|)
(Intercept) 1.115907    0.435915 2.559918 0.01046967
```

Do a usual statistical (GLIM) analysis!

Example:
10 out of 15 correct answers in a triangle test:

```
glm(t(c(10,5))~1, family=triangle,mustart=10/15)

              Estimate   Std. Error  z value    Pr(>|z|)
(Intercept) 2.321377   0.6510397 3.565646 0.00036296
```

Do a usual statistical (GLIM) analysis!

➢SensDiscrimSimple(10,15,"threeAFC")

```
      dprime      SE        P-value
[1,] 1.115907 0.435915 0.008504271
```

➢SensDiscrimSimple(10,15,"triangle")

```
      dprime      SE         P-value
[1,] 2.321377 0.6510397 0.008504271
```

Triangle, duo-trio and K-AFC as GLIMs

A practical perspective:

1. Find some GLIM-software

2. Define the psychometric link functions in the software (user defined links are usually an option)

3. Do a ”usual linear type” (GLIM) statistical analysis with an inbuilt routine to obtain:

   1. Estimates of d-primes
   2. Variances of d-primes
   3. Comparisons of different d-primes

Comparison of d-primes

Example, 3-AFC:

Sample A: 10 out of 15 correct answers

Sample B: 7 out of 15 correct answers

```
➢result<-glm(y~sample-1,family=threeAFC,data=dat,mustart=mustart)
➢estimable(result,matrix(c(1,-1),ncol=2),conf.int=0.95)

Estimate    Std. Error    Lower CI     Upper CI
0.887053   0.6072979   -0.3569403   2.131046
```

In analogy with a 2-sample t-test!

A-Not A and Same-different methods

1. Same-different protocol:
   1. NOT a GLIM!!
   2. Maximum likelihood easy in R
2. A-Not A:

$$d' = \Phi^{-1}\left(\frac{\#(\text{notA responses})}{\#(\text{notA samples})}\right) - \Phi^{-1}\left(\frac{\#(\text{notA responses})}{\#(\text{A samples})}\right)$$

$$d' = d_N - d_A$$

$$\Phi^{-1}(p_i) = d_{Sample(i)}$$

The A-Not A as a GLIM

$y_i \sim bin(1, p_i)$	*(Distribution)*
$\Phi^{-1}(p_i) = d_{Sample(i)}$	*(Probit Link)*

d-prime is a *contrast in a two-sample case:*

$$d' = d_N - d_A$$

The A-Not A as a GLIM

Example:

Not A Samples: 58 out of 100 Not A answers

A Samples: 43 out of 100 Not A answers

- glm(y~sample-1, data=dat,family=binomial(link="probit"))
- estimable(res,matrix(c(1,-1),ncol=2))

Estimate	Std. Error
0.3782676	0.1784065

Replicated Triangle, duo-trio and K-AFCs

Response for jth replication for judge i:

$$y_{ij} \sim bin(1, p_{ij})$$ *(Distribution)*

$$g^{-1}(p_{ij}) = \delta_i$$ *(Link)*

Random judge effect: $\delta_i \sim N(\delta, \sigma^2)$

Generalized linear *mixed* model!

Replicated Triangle, duo-trio and K-AFCs

- Brockhoff (2003), FQP:
  - Generalized Linear Mixed Models are very similar to beta-binomial models!
  - In both cases: the individual heterogeneity is captured by an additional dispersion parameter.
- Lot's of theory, methods and software exist for such models!

Replicated 2-AFC: an example

10 out of 15 correct answers in 5x3 2-AFC-tests:

```
➢glmmPQL(y~1, data=dat,family=twoAFC,random = ~ 1 |id)

                 Value     Std.Error
(Intercept) 0.7761957 0.8395696
```

Compare with unreplicated case:

```
➢SensDiscrimSimple(10,15,"twoAFC")

    dprime        SE
  0.6091404   0.4734123
```

Summary

1. SOME thurstonian models can be identified as GLIMs

2. Can use theory, methodology and software developed for GLIMs to
   A. Compute d-primes
   B. Compute variances of d-primes
   C. Compare different d-primes

3. An R-package with the required additionals is provided.

4. Generalized Linear *Mixed* models offer an alternative approach to the handling of replications:
   A. Methods and software can be found!
   B. Needs more clarifying research work.