



# Thurstonian models as Generalized Linear Models



Per Bruun Brockhoff

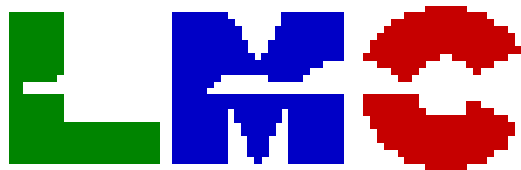
Dept. of Natural Sciences and

Centre for Advanced Food Studies(LMC)

The Royal Veterinary and Agricultural University (KVL)

Thorvaldsensvej 40, DK-1871 Frederiksberg C, Denmark

<http://www.dina.kvl.dk/~per>



- 1. General and Generalized linear models**
- 2. Triangle, duo-trio and K-AFC as GLIMs**
- 3. The R language.**
- 4. A-Not A and Same-different methods**
- 5. Replicated difference tests**
- 6. Summary**

$$y = X\beta + \varepsilon$$

or

$$y_i = \mathbf{x}_i\beta + \varepsilon_i$$

- T-tests
- Analysis of variance (ANOVA, fixed effects)
- Simple and Multiple regression analysis (MLR)
- Analysis of covariance (ANCOVA)

1. The estimated parameter values:  $\hat{\beta}$
2. The estimated uncertainty about these:

$$\text{Cov}(\hat{\beta})$$

- 1. Everything else is deduced from this!**
- 2. We have explicit formulas for computation!**

40 male and 40 female consumers rated sweetness intensity ( $y$ ) of samples with varying stimulus concentrations ( $x$ ).

A linear model with a gender-dependent intensity-stimulus relation:

$$y_i = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i + \varepsilon_i$$

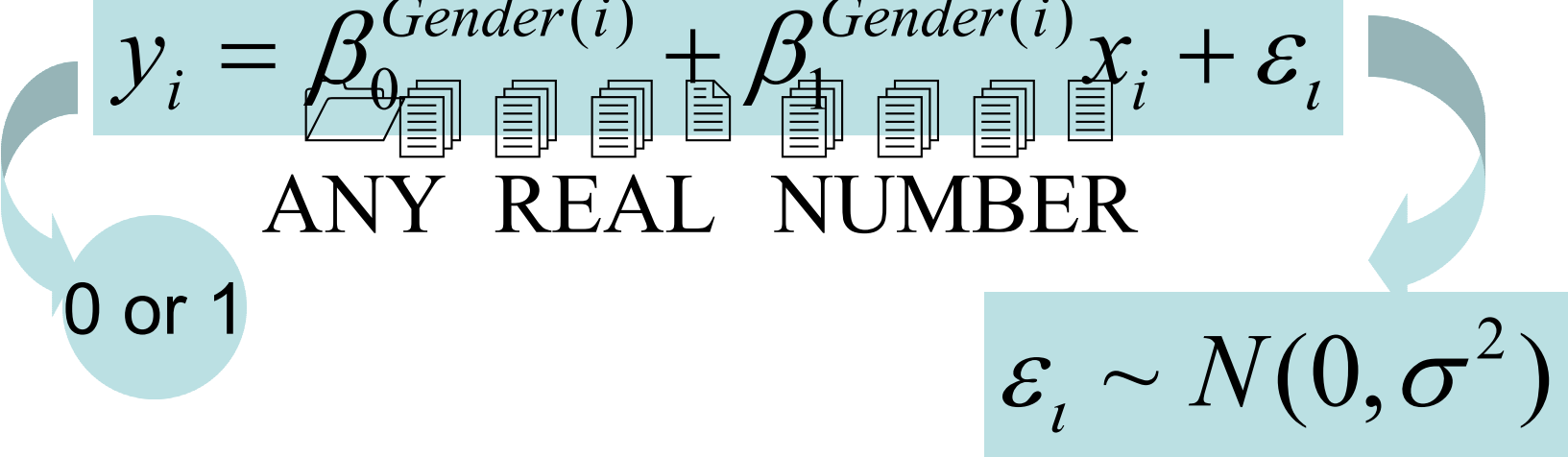
OR EQUIVALENTLY:

$$E(y_i) = \beta_0^{Gender(i)} + \beta_1^{Gender(i)} x_i$$

What if we had 80 triangle test instead?

$$y_i \sim \text{bin}(1, p_i), i = 1, \dots, 80.$$

First approach:

$$y_i = \beta_0^{\text{Gender}(i)} + \beta_1^{\text{Gender}(i)} x_i + \varepsilon_i$$


ANY REAL NUMBER

0 or 1

$$\varepsilon_i \sim N(0, \sigma^2)$$

Not natural!

What if we had 80 triangle test instead?

$$y_i \sim \text{bin}(1, p_i), i = 1, \dots, 80.$$

Second approach:





$$E y_i = \beta_0^{\text{Gender}(i)} + \beta_1^{\text{Gender}(i)} x_i$$

What if we had 80 triangle test instead?

$$y_i \sim \text{bin}(1, p_i), i = 1, \dots, 80.$$

Second approach:

$$p_i = \beta_0^{\text{Gender}(i)} + \beta_1^{\text{Gender}(i)} x_i$$

ANY REAL NUMBER



Any number between 0 and 1

Better, but not perfect!!

What if we had 80 triangle tests instead?

$$y_i \sim \text{bin}(1, p_i), i = 1, \dots, 80.$$

Final approach:

$$\log(p_i / (1 - p_i)) = \beta_0^{\text{Gender}(i)} + \beta_1^{\text{Gender}(i)} x_i$$

Logistic Regression Model:

- $y$  is binomial *distributed*
- $E(y)$  is *linked* to a linear model by the logit (log-odds) function

1. The *distribution* of  $y$   
(binomial, poisson, gamma, normal etc)
2. The *link* between  $E(y)$  and the linear model structure  
(logit, probit, log, inverse etc)

The Logistic Regression Model is a GLIM with

1. binomial *distribution*
2. logit *link*

A theoretical AND computational/practical framework for statistical analysis in many situations!

1. The estimated parameter values:  $\hat{\beta}$
2. The estimated uncertainty about these:

$$\text{Cov}(\hat{\beta})$$

- 1. Everything else is deduced from this!**
- 2. We have simple iterative methods and available software for computation!**

Find d-prime from proper psychometric function g:

$$g(d') = \hat{p}$$

Corresponding model:

$$y_i \sim \text{bin}(1, p_i) \quad (\text{Distribution})$$

$$g^{-1}(p_i) = \delta \quad (\text{Link})$$

A practical perspective:

1. Find some GLIM-software
2. Define the psychometric link functions in the software  
(user defined links are usually an option)
3. Do a "usual linear type" (GLIM) statistical analysis  
with an inbuilt routine to obtain:
  1. Estimates of d-primes
  2. Variances of d-primes
  3. Comparisons of different d-primes



<http://www.r-project.org/>

A language and environment for  
statistical computing and graphics

Similar to the S-language (Splus)

Free Software





- package provided by me along with  
this paper!

For instance, 3-AFC:

```
threeAFCg<-function(x,d) dnorm(x-d)*pnorm(x)^2
threeAFCgd<-function(x,d) -(x-d)*dnorm(x-d)*pnorm(x)^2

threeAFC<-binomial()
threeAFC$link <- "Link for the 3-AFC test"
threeAFC$linkinv <- function(eta) integrate(threeAFCg,-Inf,Inf,d=eta)$value
threeAFC$mu.eta <- function(eta) integrate(threeAFCgd,-Inf,Inf,d=eta)$value
threeAFCg2<-function(d,p) -p+threeAFC$linkinv(d)
threeAFC$linkfun <- function(mu){
if (mu>1/3) res<- uniroot(threeAFCg2,c(0,10),p=mu)$root
if (mu<=1/3)res <- 0
res
}
```

Example:

10 out of 15 correct answers in a 3-AFC test:

```
glm(t(c(10,5))~1, family=threeAFC,mustart=10/15)
```




	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	1.115907	0.435915	2.559918	0.01046967

Example:


10 out of 15 correct answers in a triangle test:

```
glm(t(c(10,5))~1, family=triangle,mustart=10/15)
```




	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	2.321377	0.6510397	3.565646	0.00036296

➤ SensDiscrimSimple(10,15,"threeAFC")



	dprime	SE	P-value
[1,]	1.115907	0.435915	0.008504271

➤ SensDiscrimSimple(10,15,"triangle")



	dprime	SE	P-value
[1,]	2.321377	0.6510397	0.008504271

A practical perspective:

1. Find some GLIM-software
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  3. Comparisons of different d-primes

Example, 3-AFC:

Sample A: 10 out of 15 correct answers

Sample B: 7 out of 15 correct answers

```
➤ result<-glm(y~sample-1,family=threeAFC,data=dat,mustart=mustart)
➤ estimable(result,matrix(c(1,-1),ncol=2),conf.int=0.95)
```



Estimate	Std. Error	Lower CI	Upper CI
0.887053	0.6072979	-0.3569403	2.131046

In analogy with a 2-sample t-test!

## 1. Same-different protocol:

1. NOT a GLIM!!

2. Maximum likelihood easy in 

## 2. A-Not A:

$$d' = \Phi^{-1}\left(\frac{\#(\text{notA responses})}{\#(\text{notA samples})}\right) - \Phi^{-1}\left(\frac{\#(\text{notA responses})}{\#(\text{A samples})}\right)$$

$$d' = d_N - d_A$$

$$\Phi^{-1}(p_i) = d_{\text{Sample}(i)}$$

$$y_i \sim \text{bin}(1, p_i) \quad (\text{Distribution})$$

$$\Phi^{-1}(p_i) = d_{\text{Sample}(i)} \quad (\text{Probit Link})$$

d-prime is a *contrast in a two-sample case*:

$$d' = d_N - d_A$$

Example:

Not A Samples: 58 out of 100 Not A answers

A Samples: 43 out of 100 Not A answers

- `glm(y~sample-1, data=dat,family=binomial(link="probit"))`
- `estimable(res,matrix(c(1,-1),ncol=2))`



Estimate	Std. Error
0.3782676	0.1784065

Response for  $j$ th replication for judge  $i$ :

$$y_{ij} \sim \text{bin}(1, p_{ij})$$

*(Distribution)*

$$g^{-1}(p_{ij}) = \delta_i$$

*(Link)*

Random judge effect:


$$\delta_i \sim N(\delta, \sigma^2)$$

**Generalized linear *mixed* model!**

- Brockhoff (2003), FQP:
  - Generalized Linear Mixed Models are very similar to beta-binomial models!
  - In both cases: the individual heterogeneity is captured by an additional dispersion parameter.
- Lot's of theory, methods and software exist for such models!

10 out of 15 correct answers in 5x3 2-AFC-tests:


➤ `glmmPQL(y~1, data=dat,family=twoAFC,random = ~ 1 |id)`



	Value	Std.Error
(Intercept)	0.7761957	0.8395696

Compare with unrepliated case:

➤ `SensDiscrimSimple(10,15,"twoAFC")`



	dprime	SE
	0.6091404	0.4734123

1. SOME thurstonian models can be identified as GLIMs
2. Can use theory, methodology and software developed for GLIMs to
  - A. Compute d-primes
  - B. Compute variances of d-primes
  - C. Compare different d-primes
3. An R-package with the required additional is provided.
4. Generalized Linear *Mixed* models offer an alternative approach to the handling of replications:
  - A. Methods and software can be found!
  - B. Needs more clarifying research work.

